An Efficient Finite Element Formulation to Analyze Waveguides with Lossy Inhomogeneous Bi-Anisotropic Materials

Luis Valor and Juan Zapata

Abstract—**In this paper a finite element formulation in terms of the magnetic field is presented for the analysis of waveguides with bianisotropic media. Such a formulation can deal with lossy inhomogeneous materials characterized by simultaneous permittivity, permeability, and cross-coupling (as in optical activity) arbitrary full tensors. The analysis takes into account arbitrary cross sections, and results in spurious-mode suppression, complex-mode computation, and the possibility of alternatively specifying the frequency or the complex propagation constant as an input parameter. In this way, many novel classes of waveguides with promising applications, such as chirowaveguides and chiroferrite-waveguides, can be analyzed. The formulation leads to a quadratic sparse eigenvalue problem which is transformed into a sparse generalized eigenvalue problem. This eigensystem is solved by the subspace method, the sparsity of the matrices being fully utilized. The proposed method has been validated by analyzing waveguides with biisotropic and bianisotropic materials. The agreement with previously published data is found to be excellent.**

I. Introduction

IN RECENT YEARS, much research effort has been devoted to the analysis of electromagnetic waves in bianisotropic and biisotropic materials. These media are characterized by linear constitutive relations which couple the electric and magnetic field vectors by four independent tensors (bianisotropic) or by four independent scalars (biisotropic). This effort is due to the wide potential applications of gyroelectric and gyromagnetic materials, to design microwave and millimeter wave circuit components such as converters, depolarizers, polarimeters, and directional couplers [1]. A chiral medium is a special case of biisotropic medium. Such material may be used to design guided wave structure named chirowaveguide.

The finite element method (FEM) is based on a spatial discretization. This allows one to handle waveguide cross section geometries which are very similar to the real structures employed in practical devices. These complex structures do not lend themselves to analytical solutions. As a consequence the FEM constitutes a promising tool to characterize this type of applications.

Manuscript received April 10, 1995; revised November 12, 1995. This work was supported by the CICYT, Spain, TIC95-0137-C02-01.

The authors are with the Grupo de Electromagnetismo Aplicado y Microondas (GEAM), Departamento de Electromagnetismo y Teoría de Circuitos. Universidad Politécnica de Madrid, ETSI Telecomunicación, Ciudad Universitaria s/n, 28040 Madrid, Spain.

Publisher Item Identifier S 0018-9480(96)01451-2.

In spite of that, there is at present no available FEM formulation, useful for the two–dimensional (2-D) analysis of waveguides with bianisotropic media.

Svedin [2], [3] employs a six-component vector finite element formulation which permits the analysis of media characterized with cross-coupling scalars.

In this paper a formulation is proposed to solve waveguides with bi-anisotropic materials. Such bianisotropy is represented by a full 6 × 6 matrix. The proposed three component vector formulation has the capability of handling losses and the ability to compute complex modes. This is accomplished by specifying the frequency as an input parameter and solving for the complex propagation constant as the eigenvalue. Spurious solution appearance is suppressed by employing an edge element which, besides, enables the analysis of structures with reentrant corners.

This formulation leads to a sparse complex nonhermitic quadratic eigenvalue problem which is transformed into a sparse complex nonhermitic generalized one [4]. Such eigensystem is solved efficiently by a method based on the subspace iteration algorithm which fully makes use of the sparsity of the matrices. Versatility and accuracy of the proposed finite element formulation are assessed by analyzing different bianisotropic waveguides. The results obtained show excellent agreement with previously available data.

II. Theoretical Analysis

Any material may be characterized, from the macroscopic point of view, by means of the following set of constitutive relations [5] for the time harmonic excitation

$$\begin{pmatrix} \vec{D} \\ \vec{B} \end{pmatrix} = [M]\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \tag{1}$$

where $\vec{D}, \vec{B}, \vec{E}, \vec{H}$ represent, respectively, the electric displacement, magnetic induction, electric, and magnetic field

$$[M] = \begin{bmatrix} \varepsilon_0[\varepsilon] & [\xi] \\ [\varsigma] & \mu_0[\mu] \end{bmatrix} \tag{2}$$

is the characteristic bianisotropic 6 × 6 tensor where ε_0 and μ_0 are the permittivity and permeability of free space and $[\varepsilon], [\mu], [\xi], [\varsigma]$ the relative permittivity, the relative permeability and the two cross-coupling tensors, respectively. Such tensors are written, in a rectangular Cartesian coordinate system, as 3 × 3 complex matrices in the form

$$\begin{aligned}\bar{\varepsilon} = [\varepsilon] &= \begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix}\\ \bar{\mu} = [\mu] &= \begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{yx} & \mu_{yy} & \mu_{yz}\\ \mu_{zx} & \mu_{zy} & \mu_{zz}\end{bmatrix}\\ \bar{\xi} = [\xi] &= \begin{bmatrix}\xi_{xx} & \xi_{xy} & \xi_{xz}\\ \xi_{yx} & \xi_{yy} & \xi_{yz}\\ \xi_{zx} & \xi_{zy} & \xi_{zz}\end{bmatrix}\\ \bar{\varsigma} = [\varsigma] &= \begin{bmatrix}\varsigma_{xx} & \varsigma_{xy} & \varsigma_{xz}\\ \varsigma_{yx} & \varsigma_{yy} & \varsigma_{yz}\\ \varsigma_{zx} & \varsigma_{zy} & \varsigma_{zz}\end{bmatrix}.\end{aligned} \tag{3}$$

Two interesting particular cases to be pointed out occur when $[\xi]$ and $[\varsigma]$ are both null and when the characteristic bianisotropic tensor becomes real. In that case the bianisotropic media become anisotropic and biisotropic media, respectively.

Let us consider a waveguide of arbitrary cross-sectional shape Ω in the x-y plane, uniform along the direction of propagation (z-axis) and filled with lossy inhomogeneous and bianisotropic material as described by (1) and (2). Its boundary Γ is a combination of a perfect electric conductor Γ_1 and a perfect magnetic conductor Γ_2 (Fig. 1).

The source-free Maxwell equations can be written as

$$\begin{aligned}\nabla \times \vec{E} &= -j\omega\vec{B}\\ \nabla \times \vec{H} &= j\omega\vec{D}\end{aligned} \tag{4}$$

where ω is the angular frequency. It is assumed that the electromagnetic field in the waveguide varies as $e^{(j\omega t-\gamma z)}$, where $\gamma = \alpha + j\beta$ is the complex propagation constant, and α, β are the attenuation and phase constants, respectively.

From (1), (2), and (4) the bianisotropic Helmholtz equation for the magnetic field is found to be

$$\begin{aligned}\nabla \times \bar{e}\nabla \times \vec{H} - j\omega\nabla \times \bar{e}\bar{\xi}\vec{H} + j\omega\bar{\varsigma}\bar{e}\nabla \times \vec{H}\\ + \omega^2\bar{\xi}\bar{e}\bar{\varsigma}\vec{H} - k_0^2\bar{\mu}\vec{H} = 0\end{aligned} \tag{5}$$

where

$$\bar{e} = \bar{\varepsilon}^{-1} = \begin{bmatrix}e_{xx} & e_{xy} & e_{xz}\\ e_{yx} & e_{yy} & e_{yz}\\ e_{zx} & e_{zy} & e_{zz}\end{bmatrix} \tag{6}$$

and k_0 is the free space wavenumber.

The boundary conditions are satisfied by enforcing the following equations

$$\begin{aligned}\hat{n} \times (\bar{e}\nabla \times \vec{H} - j\omega\bar{e}\bar{\xi}\vec{H}) = 0 \quad &\text{on } \Gamma_1\\ \hat{n} \times \vec{H} = 0 \quad &\text{on } \Gamma_2\end{aligned} \tag{7}$$

where $\hat{n}$ is a unit vector in the same plane of Ω, normal to the boundary Γ and directed outwards (Fig. 1).

Considering trial functions $\vec{H}$ and test functions $\vec{w}$ in an admissible space [6], and applying vectorial identities to the expression

$$\begin{aligned}\int_\Omega \vec{w}(\nabla \times \bar{e}\nabla \times \vec{H} - j\omega\nabla \times \bar{e}\bar{\xi}\vec{H}\\ + j\omega\bar{\varsigma}\bar{e}\nabla \times \vec{H} + \omega^2\bar{\xi}\bar{e}\bar{\varsigma}\vec{H} - k_0^2\bar{\mu}\vec{H})d\Omega = 0\end{aligned} \tag{8}$$

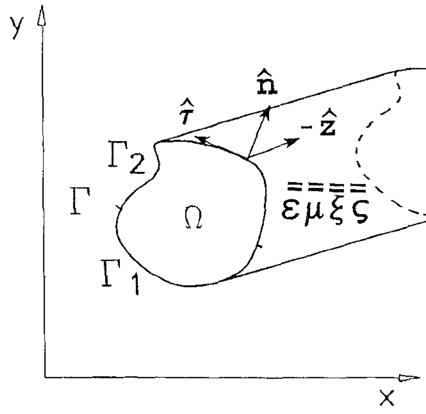

Fig. 1. General waveguide geometry

the following weak formulation of the boundary-value problem can be derived

$$\begin{aligned}\Im(\vec{H}, \vec{w}) = &-\int_{\Gamma_1} \vec{w}[\hat{n} \times (\bar{e}\nabla \times \vec{H} - j\omega\bar{e}\bar{\xi}\vec{H})]d\Gamma_1\\ &+ \int_{\Gamma_2} [\bar{e}\nabla \times \vec{H} - j\omega\bar{e}\bar{\xi}\vec{H}] \cdot [\hat{n} \times \vec{w}]d\Gamma_2.\end{aligned} \tag{9}$$

After splitting the trial functions $\vec{H}$, the test functions $\vec{w}$ and the operator ∇ into their transverse and axial parts, the bilinear form $\Im(\vec{H}, \vec{w})$ takes the expression

$$\begin{aligned}&\Im(\vec{H}, \vec{w})\\ &= \int_\Omega \{(\nabla_t \times \vec{w}_t)\hat{z}\\ &\quad \times [\bar{e}'\hat{z}\nabla_t \times \vec{H}_t - \bar{e}'\nabla_t H_z - j\omega\bar{\varphi}\hat{z}H_z\\ &\qquad + (-\gamma\bar{e}' - j\omega\bar{\varphi})\vec{H}_t]\\ &\quad + \nabla_t w_z[-\bar{e}'\hat{z}\nabla_t \times \vec{H}_t + \bar{e}'\nabla_t H_z\\ &\qquad + (\gamma\bar{e}' + j\omega\bar{\varphi}')\vec{H}_t + j\omega\bar{\varphi}'\hat{z}H_z]\\ &\quad + \vec{w}_t[(\gamma\bar{e}' + j\omega\bar{\psi}')\hat{z}\nabla_t \times \vec{H}_t + (-\gamma\bar{e}' - \bar{\psi}')\nabla_t H_z\\ &\qquad + (-\gamma^2\bar{e}' - j\omega\gamma\bar{\varphi}' - j\omega\gamma\bar{\psi}' + \omega^2\bar{\chi} - k_0^2\bar{\mu})\vec{H}_t\\ &\qquad + (-j\omega\gamma\bar{\varphi}' + \omega^2\bar{\chi} - k_0^2\bar{\mu})\hat{z}H_z]\\ &\quad + w_z\hat{z}[j\omega\bar{\psi}\hat{z}\nabla_t \times \vec{H}_t - j\omega\bar{\psi}'\nabla_t H_z\\ &\qquad + (-j\omega\gamma\bar{\psi}' + \omega^2\bar{\chi} - k_0^2\bar{\mu})\vec{H}_t\\ &\qquad + (\omega^2\bar{\chi} - k_0^2\bar{\mu})\hat{z}H_z]\}d\Omega\end{aligned} \tag{10}$$

where

$$\begin{aligned}\bar{e}' &= \begin{bmatrix}e_{yy} & -e_{yx} & e_{yz}\\ -e_{xy} & e_{xx} & -e_{xz}\\ e_{zy} & -e_{zx} & e_{zz}\end{bmatrix}\\ \bar{\chi} = \bar{\varsigma}\bar{e}\bar{\xi} &= \begin{bmatrix}\chi_{xx} & \chi_{xy} & \chi_{xz}\\ \chi_{yx} & \chi_{yy} & \chi_{yz}\\ \chi_{zx} & \chi_{zy} & \chi_{zz}\end{bmatrix}\\ \bar{\varphi} = \bar{e}\bar{\xi} &= \begin{bmatrix}\varphi_{xx} & \varphi_{xy} & \varphi_{xz}\\ \varphi_{yx} & \varphi_{yy} & \varphi_{yz}\\ \varphi_{zx} & \varphi_{zy} & \varphi_{zz}\end{bmatrix}\end{aligned}$$

$$\bar{\psi} = \bar{\varsigma}\bar{e} = \begin{bmatrix} \psi_{xx} & \psi_{xy} & \psi_{xz} \\ \psi_{yx} & \psi_{yy} & \psi_{yz} \\ \psi_{zx} & \psi_{zy} & \psi_{zz} \end{bmatrix}$$

$$\bar{\varphi}' = \begin{bmatrix} \varphi_{yx} & \varphi_{yy} & \varphi_{yz} \\ -\varphi_{xx} & -\varphi_{xy} & -\varphi_{xz} \\ \varphi_{zx} & \varphi_{zy} & \varphi_{zz} \end{bmatrix}$$

$$\bar{\psi}' = \begin{bmatrix} \psi_{yx} & -\psi_{xx} & \psi_{xz} \\ \psi_{yy} & -\psi_{xy} & \psi_{yz} \\ \psi_{zy} & -\psi_{zx} & \psi_{zz} \end{bmatrix}$$

and

$$\begin{aligned} \vec{H} &= \vec{H}_t + H_z\hat{z} \\ \vec{w} &= \vec{w}_t + w_z\hat{z} \\ \nabla &= \nabla_t - \gamma\hat{z} \\ \nabla_t &= \hat{x}\partial/\partial x + \hat{y}\partial/\partial y. \end{aligned} \tag{11}$$

By imposing homogeneous and natural boundary conditions of the form

$$\begin{aligned} &\hat{n} \times \vec{w} = 0 \quad \text{on } \Gamma_2 \\ &\hat{n} \times (\bar{e}\nabla \times \vec{H} - j\omega\bar{e}\bar{\xi}\vec{H}) = 0 \quad \text{on } \Gamma_1 \end{aligned} \tag{12}$$

respectively, (9) reduces to

$$\Im\left(\vec{H}, \vec{w}\right) = 0 \tag{13}$$

$\vec{H}$ and $\vec{w}$ belong to the same space of functions and are smooth enough for the integrals appearing in (13) to be well defined.

III. Finite Element Formulation

In this paper, triangular hybrid vector elements which combine Lagrangian basis functions for the longitudinal component with edge elements for the transverse ones, as those proposed in [7] will be employed. The three components of the magnetic field are discretized on each element according to

$$\begin{bmatrix} \{H_p\}_e \\ \{H_q\}_e \\ \{H_z\}_e \end{bmatrix} = \begin{bmatrix} \langle 0\rangle & \langle T_p(i,j)\rangle \\ \langle 0\rangle & \langle T_q(i,j)\rangle \\ \langle N_i\rangle & \langle 0\rangle \end{bmatrix} \begin{bmatrix} \{H_z\}_e \\ \{H_t\}_e \end{bmatrix} \tag{14}$$

where $\langle N_i\rangle$ are the first-degree Lagrange polynomials in the reference space (p, q) and

$$\begin{aligned} \vec{T}(i,j) &= N_i\nabla N_j - N_j\nabla N_i \\ &= T_p(i,j)\,\vec{p} + T_q(i,j)\,\vec{q} \end{aligned} \tag{15}$$

is the vectorial basis function for edge (i, j).

Making both trial and test functions be the same, the discretised equation in (10) can be expressed as

$$(\gamma^2[A] + \gamma[B] + [C] + \omega^2[D])\{\tilde{H}\} = 0 \tag{16}$$

with

$$\{\tilde{H}\} = \begin{Bmatrix} \{H_z\} \\ \{H_t\} \end{Bmatrix}$$

where either the angular frequency or the complex propagation constant, may be considered as an eigenvalue. In this paper the complex propagation constant is chosen as an eigenvalue, in order to analyze structures which support complex modes or include losses. Additionally, the z-component of the magnetic field, H_z is replaced by $H_z = H_z'\cdot\gamma$. With this transformation, a substantial simplification of the eigensystem is obtained when the proposed formulation is applied to anisotropic media in which both $[\varepsilon]$ and $[\mu]$ tensors can be expressed as $[\varepsilon] = [\varepsilon_{tt}] + \varepsilon_{zz}\hat{z}\hat{z}$ and $[\mu] = [\mu_{tt}] + \mu_{zz}\hat{z}\hat{z}$, where $[\varepsilon_{tt}], [\mu_{tt}]$ are two-by-two tensors [4]. After these manipulations, (16) can be written in the following form

$$(\gamma^2[M_1] + \gamma[M_2] + [M_3])\{H\} = 0 \tag{17}$$

where

$$\{H\} = \begin{Bmatrix} \{H_z'\} \\ \{H_t\} \end{Bmatrix}$$

and the $[M_i]$ matrices, shown in (18) at the bottom of the page with $[T_i]$ as given in Appendix. Notice that M_3 is always a singular matrix.

By setting

$$[K]' = \begin{bmatrix} [0] & [I] \\ [M_3] & [M_2] \end{bmatrix} \quad [M]' = \begin{bmatrix} [I] & [0] \\ [0] & -[M_1] \end{bmatrix}$$

and introducing an unknown vector $\bar{H} = \langle\langle \bar{H}_z\rangle\ \langle \bar{H}_t\rangle\rangle^T$, with the superscript T denoting transposition, the sparse quadratic eigenvalue problem (17) can be reduced to a generalized eigensystem

$$[K]'\{X\}' - \gamma[M]'\{X\}' = 0 \tag{19}$$

where

$$\{X\}' = \begin{Bmatrix} \{H\} \\ \{\bar{H}\} \end{Bmatrix}. \tag{20}$$

Equations in (19) can be reordered as described in [4] to transform this eigensystem into a new sparse generalized one

$$[K]\{X\} - \gamma[M]\{X\} = 0 \tag{21}$$

$$\begin{aligned} [M_1] &= \sum_e \begin{bmatrix} [T_8] - [T_{11}] - [T_{26}] + [T_{29}] & [T_5] - [T_{27}] \\ -[T_7] - [T_{23}] & -[T_6] \end{bmatrix} \\ [M_2] &= \sum_e \begin{bmatrix} [T_{17}] & -[T_3] - [T_{13}] + [T_{16}] + [T_{25}] + [T_{28}] \\ -[T_9] - [T_{12}] - [T_{15}] - [T_{19}] + [T_{24}] & [T_2] - [T_4] - [T_{20}] - [T_{21}] \end{bmatrix} \\ [M_3] &= \sum_e \begin{bmatrix} [0] & [0] \\ [0] & [T_1] - [T_{10}] - [T_{14}] + [T_{18}] + [T_{22}] \end{bmatrix} \end{aligned} \tag{18}$$

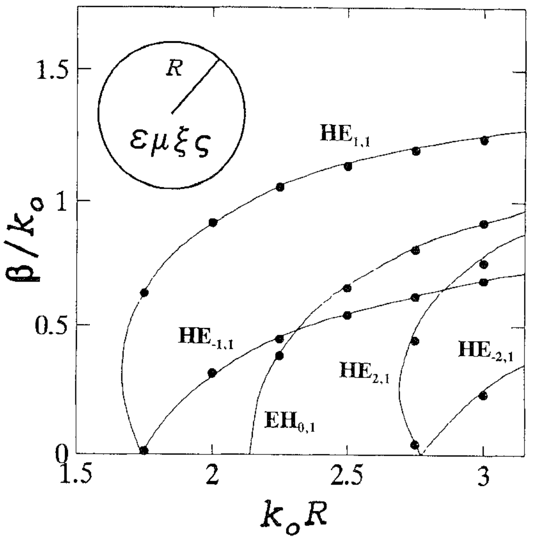

Fig. 2 Normalized propagation constant versus k_0R for a circular chirowaveguide of radius R filled with a material characterized by $\varepsilon = 1.1419$, $\mu = 1$ and $\xi = -\varsigma = j\mu_0\zeta_c$, $\zeta_c = 1$ mS (• • • •) this work. (————) [2]

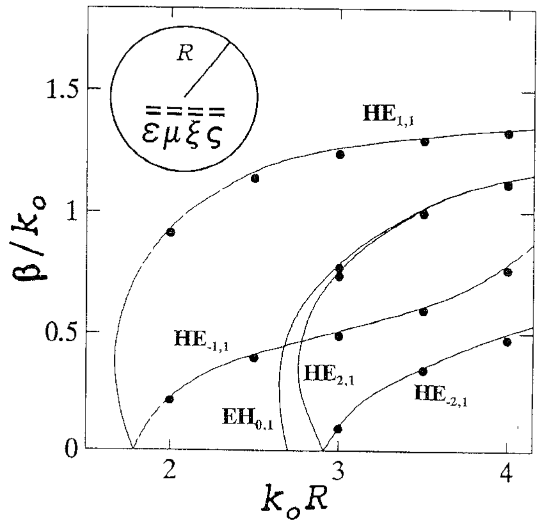

Fig. 3. Normalized propagation constant versus k_0R for a circular waveguide of radius R filled with bianisotropic material (• • • •) this work. (————) [10].

in which the sparse properties of the matrices $[K], [M]$ are equivalent to those in the original quadratic eigensystem (17).

The final eigensystem (21) involves singular, sparse and, in general, complex matrices which are neither hermitic nor symmetrical. To solve (21), an algorithm based on the subspace iteration method [4], [8], [9] has been developed in which the sparsity of the matrices is fully utilized.

IV. Numerical Examples

This section presents some numerical results in order to validate the proposed method by comparison with published data for various waveguides [2], [10].

First, the constitutive relations used in [2], [10]

$$\begin{pmatrix} \vec{D} \\ \vec{H} \end{pmatrix} = \begin{bmatrix} [\varepsilon_p] & [X^-] \\ [-X^+] & [\mu_p]^{-1} \end{bmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \tag{22}$$

are not the same as those utilized in this paper as given in (1), (2). The equivalence of both sets of parameters was derived by Lindell *et al.* [11], [12] for scalar values. In a similar way, the equivalence for tensors is found to be

$$\begin{aligned} \mu_0[\mu] &= [\mu_p] \\ [\varsigma] &= [\mu_p][X^+] \\ [\xi] &= [X^-][\mu_p] \\ \varepsilon_0[\varepsilon] &= [\varepsilon_p] + [X^-][\mu_p][X^+]. \end{aligned} \tag{23}$$

After this marginal note, we are ready to show some results. In Fig. 2, the complex propagation constants for the fundamental and higher order modes of a circular chirowaveguide of radius R are shown. This chiral medium is characterized by $\varepsilon_p = \varepsilon_0, \mu_p = \mu_0$ and $X^- = X^+ = j\zeta_c$ with the chirality admittance $\zeta_c = 1$ mS [2]. These parameters are transformed into a relative permittivity $\varepsilon = 1.1419$, a relative permeability $\mu = 1$ and cross-coupling factors $\xi = -\varsigma = j\mu_0\zeta_c$. In the figure, solid lines represent the results in [2] and dark circles the computed values obtained by the present approach. They are found to agree perfectly.

Fig. 3 displays the complex propagation constants for the same geometry but with a medium which is, according to (23), characterized by

$$[\varepsilon] = \begin{bmatrix} 1.099 & -j0.043 & 0 \\ j0.043 & 1.099 & 0 \\ 0 & 0 & 1.142 \end{bmatrix}$$

$$[\mu] = \begin{bmatrix} 0.7 & -j0.3 & 0 \\ j0.3 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\xi] = \mu_0 \begin{bmatrix} j0.7 & 0.3 & 0 \\ -0.3 & j0.7 & 0 \\ 0 & 0 & j \end{bmatrix} \times 10^{-3}$$

$$[\varsigma] = \mu_0 \begin{bmatrix} -j0.7 & -0.3 & 0 \\ 0.3 & -j0.7 & 0 \\ 0 & 0 & j \end{bmatrix} \times 10^{-3}.$$

The results obtained by the present method are drawn with dark circles. They have been compared with those available in [10], represented with solid lines, showing excellent agreement.

V. Conclusion

A finite element formulation, based on the three components of the magnetic field, is proposed for analyzing waveguides with bianisotropic materials. Such formulation has the capability to handle simultaneously the permittivity, permeability, and cross-coupling tensors that may be arbitrarily full. Its ability to compute free-spurious complex solutions gives way to the analysis of lossy structures and complex modes. The proposed formulation leads to a sparse complex generalized eigenvalue problem with matrices which, in general, are neither symmetric nor hermitic. Such an eigensystem has been efficiently solved, taking full advantage of the matrix sparsity, by implementing a method based on the subspace iteration algorithm. Waveguides with biisotropic and bianisotropic materials have been analyzed to validate the proposed method. The results obtained show an excellent agreement with previously available data.

APPENDIX
THE EXPLICIT FORM OF SUBMATRICES $[T_i]$

The form of submatrices of $[T_i]$ in the text are given by

$$[T_1] = \iint_e e_{zz}[A]^T[A]dpdq$$

$$[T_2] = \iint_e [T]^T[\bar{e}'_{s1}][A]dpdq$$

$$[T_3] = \iint_e [D]^T[\bar{e}'_{s1}][A]dpdq$$

$$[T_4] = \iint_e [A]^T[\bar{e}'_{s2}][T]dpdq$$

$$[T_5] = \iint_e [D]^T[\bar{e}'_{tt}][T]dpdq$$

$$[T_6] = \iint_e [T]^T[\bar{e}'_{tt}][T]dpdq$$

$$[T_7] = \iint_e [T]^T[\bar{e}'_{tt}][D]dpdq$$

$$[T_8] = \iint_e [D]^T[\bar{e}'_{tt}][D]dpdq$$

$$[T_9] = \iint_e [A]^T[\bar{e}'_{s2}][D]dpdq$$

$$[T_{10}] = \iint_e k_0^2[T]^T[\bar{\mu}_{tt}][T]dpdq$$

$$[T_{11}] = \iint_e k_0^2\mu_{zz}[N]^T[N]dpdq$$

$$[T_{12}] = \iint_e k_0^2[T]^T[\bar{\mu}_{s1}][N]dpdq$$

$$[T_{13}] = \iint_e k_0^2[N]^T[\bar{\mu}_{s2}][T]dpdq$$

$$[T_{14}] = \iint_e j\omega[A]^T[\bar{\varphi}_{s2}][T]dpdq$$

$$[T_{15}] = \iint_e j\omega\varphi_{zz}[A]^T[N]dpdq$$

$$[T_{16}] = \iint_e j\omega[D]^T[\bar{\varphi}'_{tt}][T]dpdq$$

$$[T_{17}] = \iint_e j\omega[D]^T[\bar{\varphi}'_{s1}][N]dpdq$$

$$[T_{18}] = \iint_e j\omega[T]^T[\bar{\psi}_{s1}][A]dpdq$$

$$[T_{19}] = \iint_e j\omega[T]^T[\bar{\psi}'_{tt}][D]dpdq$$

$$[T_{20}] = \iint_e j\omega[T]^T[\bar{\varphi}'_{tt}][T]dpdq$$

$$[T_{21}] = \iint_e j\omega[T]^T[\bar{\psi}'_{tt}][T]dpdq$$

$$[T_{22}] = \iint_e \omega^2[T]^T[\bar{X}_t t][T]dpdq$$

$$[T_{23}] = \iint_e j\omega[T]^T[\bar{\varphi}'_{s1}][N]dpdq$$

$$[T_{24}] = \iint_e \omega^2[T]^T[\bar{X}_{s1}][N]dpdq$$

$$[T_{25}] = \iint_e j\omega\psi_{zz}[N]^T[A]dpdq$$

$$[T_{26}] = \iint_e j\omega[N]^T[\bar{\psi}'_{s2}][D]dpdq$$

$$[T_{27}] = \iint_e j\omega[N]^T[\bar{\psi}'_{s2}][T]dpdq$$

$$[T_{28}] = \iint_e \omega^2[N]^T[\bar{X}_{s2}][T]dpdq$$

$$[T_{29}] = \iint_e \omega^2 X_{zz}[N]^T[N]dpdq$$

where

$$[\bar{\mu}_{s1}] = \begin{Bmatrix}\mu_{xz}\\ \mu_{yz}\end{Bmatrix} \qquad [\bar{e}'_{s1}] = \begin{Bmatrix}e_{yz}\\ -e_{xz}\end{Bmatrix}$$

$$[\bar{\mu}_{s2}] = \langle \mu_{zx} \quad \mu_{zy}\rangle \qquad [\bar{e}'_{s2}] = \langle e_{zy} \quad -e_{zx}\rangle$$

$$[\bar{\mu}_{tt}] = \begin{bmatrix}\mu_{xx} & \mu_{xy}\\ \mu_{yx} & \mu_{yy}\end{bmatrix} \qquad [\bar{e}'_{tt}] = \begin{bmatrix}e_{yy} & -e_{yx}\\ -e_{xy} & e_{xx}\end{bmatrix}$$

$$[\bar{\varphi}'_{s1}] = \begin{Bmatrix}\varphi_{yz}\\ -\varphi_{xz}\end{Bmatrix} \qquad [\bar{\psi}_{s1}] = \begin{Bmatrix}\psi_{xz}\\ \psi_{yz}\end{Bmatrix}$$

$$[\bar{\varphi}_{s2}] = \langle \varphi_{zx} \quad \varphi_{zy}\rangle \qquad [\bar{\psi}'_{s2}] = \langle \psi_{zy} \quad -\psi_{zx}\rangle$$

$$[\bar{\varphi}'_{tt}] = \begin{bmatrix}\varphi_{yx} & \varphi_{yy}\\ -\varphi_{xx} & -\varphi_{xy}\end{bmatrix} \qquad [\bar{\psi}'_{tt}] = \begin{bmatrix}\psi_{xy} & -\psi_{xx}\\ \psi_{yy} & -\psi_{yx}\end{bmatrix}$$

$$[\bar{X}_{s1}] = \begin{Bmatrix}X_{xz}\\ X_{yz}\end{Bmatrix} \qquad [\bar{X}_{s2}] = \langle X_{zx} \quad X_{zy}\rangle$$

$$[T] = \begin{bmatrix}\langle T_p\rangle\\ \langle T_q\rangle\end{bmatrix} \qquad [\bar{X}_{tt}] = \begin{bmatrix}X_{xx} & X_{xy}\\ X_{yx} & X_{yy}\end{bmatrix}$$

$$[N] = \langle N_i\rangle \qquad [D] = \begin{bmatrix}\langle \frac{\partial N_i}{\partial p}\rangle\\ \langle \frac{\partial N_i}{\partial q}\rangle\end{bmatrix}$$

$$[A] = \left\langle \frac{\partial T_q}{\partial p} - \frac{\partial T_p}{\partial q}\right\rangle.$$

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Luis Valor was born in Madrid, Spain, in 1967. He received the Ingeniero degree in April 1992 and the Ph.D. degree in May 1995, both from the Universidad Politécnica de Madrid, Spain.

Since 1991 he has been with the Grupo de Electromagnetismo Aplicado y Microondas de Madrid with a fellowship. His research interest include electromagnetic propagation in inhomogeneous-anisotropic waveguides structures and computer methods in electromagnetics.

Juan Zapata received the Ingeniero de Telecommunicación degree in 1970 and the Ph.D. degree in 1974, both from the Universidad Politécnica de Madrid, Spain.

Since 1970 he has been with the Grupo de Electromagnetismo Aplicado y Microondas at the Universidad Politécnica de Madrid, where he became an Assistant Professor in 1970, Associate Professor in 1975, and Professor in 1983. He has been engaged in research on microwave active circuits and interactions of electromagnetic fields with biological tissues. His current research interest include computer-aided design for microwave passive circuits and numerical methods in electromagnetism.